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The Impact of the Exchange Fees on Impermanent Loss of Liquidity Providers for Conservative Automated Market Makers

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Abstract
Automated Market Makers (AMMs) with a conservative function, such as Uniswap, Balancer, Curve and others, are an integral part of decentralised finance. This article examines the effect of the exchange fees on the divergence losses of the automated market-making systems in public blockchain networks. The study consists of several parts: theoretical background, detailed description of the exchange mechanics, the derivation of explicit formulas, the results of modelling using the hyperparameters of pools from the Ethereum network and the analysis of the proposed approach using historical data. For the first time, the obtained closed formulas (Uniswap, Balancer) and modelling results (Uniswap, Balancer, Curve) indicate the presence of the *impermanent gain* for liquidity providers in the case of non-zero fees when the trading volume does not exceed a certain amount. The results indicate that the proposed methodology significantly affects the definition of 'impermanent loss of a liquidity provider' widely used in the blockchain community since there can always be a profitable range of values. As a practical part of the study, statistics on the share of trades with the effect of impermanent gain in Ethereum pools are provided, and the approach for managing the fee rate is considered during this observation. Explicit relationships for mostly used AMMs with non-zero trading fees are derived. The article may be useful for both practitioners and researchers in the field of decentralised finance seeking a deeper understanding of the dynamics of automated market-making in an ever-changing DeFi environment.

Keywords: *Decentralised Finance, DeFi, Decentralised Exchange, Constant Function Market Makers, AMM, Impermanent Loss, Impermanent Gain, Uniswap, Curve*

JEL Classifications: *D01, D40, D49*

1. Introduction

The development of blockchain technologies started in 2008 with the famous Satoshi Nakomoto's paper [1]. The idea behind it was to develop a decentralised and transparent ledger capable of recording transactions, which is why this technology initially found its primary application in digital currency. In the following years, blockchain technologies attracted significant attention from the community and developers, expanding beyond just digital currencies. The evolution of blockchain with the groundbreaking technology of smart contracts in Ethereum [2] enabled the implementation of complex business logic and applications within decentralised networks. One example of such innovation is Automated Market Makers (AMMs), which utilise mathematical formulas to model markets and provide liquidity for various financial assets [3].

In modern decentralised finance (DeFi), AMMs are crucial, serving as intermediaries between liquidity providers and ordinary users looking to exchange their tokens. Unlike order book trading systems [3], AMMs utilise liquidity pools and algorithmic pricing mechanisms to enable users to trade assets directly, avoiding a centralised order-matching process. Moreover, AMM systems have no spread for buying and selling. However, each unit of exchange volume always leads to a slight price slippage. In DeFi, trading systems based on AMM frequently concentrate way more liquidity than classic systems based on order books.

Over the past few years, research and development in AMMs have advanced rapidly, paving the way for numerous areas of exploration: optimising swap invariant functions [4], analysing impermanent loss and risk management [5, 6], researching user strategies [7] and much more.

This article proposes to examine the mathematical aspects of the AMM microeconomics theory of impermanent losses incurred by liquidity providers with the influence of exchange fees. The analysis of impermanent loss is a critical task. In Loesch et al. [8], researchers present a statistical quantitative result that 49.5% of liquidity providers in **Uniswap v3** have negative returns, indicating a vast field for optimisation of AMM operations regardless of the specific invariant.

In most practical implementations of AMM, the constancy of liquidity invariant after a trade is not observed due to transaction fees. Studying this phenomenon is an interesting challenge. However, it has limited coverage in the existing literature. In Xue et al. and Aigner and Dhaliwal [3, 9], the impact of varying invariants is ignored, leading to inaccurate formulations for calculating impermanent losses. In Angeris et al. [10], the case of an increasing invariant is considered; however, explicit relationships were not derived, only the estimation that is consistent with this study's findings.

It is worth noting that among modern types of AMM systems, there is an option of the conservative function **Uniswap v3** with concentrated liquidity [7], where the collected fees do not contribute to pool liquidity. Therefore, the formula for impermanent losses from Xu et al. [3] is correct. However, in this article, we will skip over this scenario and focus on the traditional versions of **Uniswap** [11], **Balancer** [3] and **Curve** [12], which we will describe in more detail later. The analysis of fee structure for invariants with concentrated liquidity is also a challenging task subject to future research.

The relevance of this study is to expand our fundamental understanding of how automated market-making systems with conservative functions really work. The presented findings could be used to optimise the actions of liquidity providers or fee adjustments based on market conditions. As the Central Bank Digital Currencies (CBDCs) and stablecoins are evolving, a detailed analysis of the microeconomic features of AMMs, such as the effect of impermanent loss described in this article, is highly demanded.

This work is divided into six parts. The current part provides a brief introduction and review of the related work. Section 2 outlines the primary mathematical concepts of automatic market makers. Section 3 presents the theoretical analysis of impermanent losses, considering the influence of the trading fee parameters and visualisations of results. This part mentions exceptional cases of **Uniswap**, **Balancer** and **Curve** invariants, and the simplest asymptotic estimation for the impermanent gain range is derived. Sections 4 and 5 discuss the practical recommendations and constraints of the proposed analytical methodology. Lastly, the conclusions of the current research are discussed.

2. Theoretical Background

In analysing the impermanent losses of liquidity providers associated with AMMs, it is crucial to take into account dependence on several factors such as liquidity, trading volumes and the particular implementation of the exchange protocol. This chapter is devoted to the theoretical foundations of automated conservative market-making systems.

2.1. Automated Market Makers

An automated market-making system is a model that utilises a fixed mathematical relationship between assets, enabling stability and predictability in pricing. Numerous articles [13, 14] were published to explore the mathematical implications of the axiomatic properties of automated market-making systems. For the purpose of the current article, we introduce the following simplified formalisation.

Without loss of generality, we assume that the automated market-making system operates with two assets, whose volumes are denoted by $x > 0$ and $y > 0$. The **conservative function** [3] (otherwise the **invariant** [10] or **trading function** [15]) is concave, increasing in each argument and piecewise differentiable function $I: {R_+}^2 \to R_+$. In practical implementation, it is additionally assumed that I is a homogeneous function. For further unification, it is more convenient to move on to considering an implicitly defined relationship that describes the dependence of the system behaviour,

$$F\big(x, y, I(x, y)\big) = F(x, y, K) = 0, \tag{1}$$

where the **liquidity** factor K is introduced, so that homogeneity of the degree q of the conservative function is observed:

$$F(sx, sy, sK) = s^q F(x, y, K). \tag{2}$$

Well-known examples of automated market-making systems are **Uniswap** [11], **Balancer** [3] and **Curve** [12]. Comparative visualisations of exchange invariants are presented in Figure 1. Such conservative functions for mentioned AMMs are respectively expressed as

$$F_u(x, y, K) = xy - K^2, \tag{3}$$

$$F_b(x, y, K) = x^{w_x} y^{w_y} - K^{w_x + w_y}, \tag{4}$$

$$F_c(x, y, K) = 4A(x + y) - (4A - 1)K - \frac{K^3}{4xy}. \tag{5}$$

Uniswap (3) is a conservative constant product market maker, a generalisation of which is the **Balancer**'s (4) geometric mean market maker [16]. **Curve** invariant is structurally constructed as a hybrid of a constant sum and product function. It is cumbersome to express the function $I_{curve}(x, y)$ explicitly (it requires solving a cubic polynomial equation); however, the relation for two traded assets in a liquidity pool is expressed implicitly in (5), where the parameter A has the meaning of amplification (or weight) coefficient between the constant sum and constant product invariants. The introduced functions F_u, F_b and F_c are homogeneous of degrees 2, $w_x + w_y$ and 1, respectively.

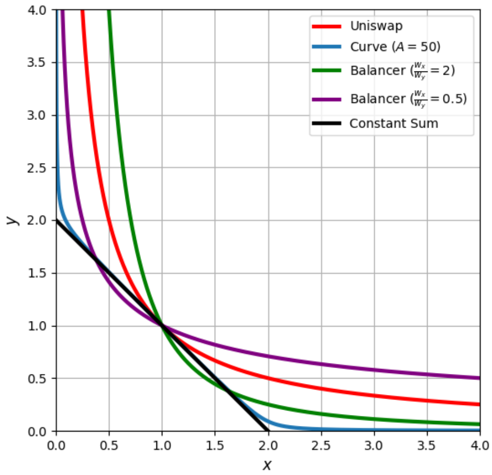

Figure 1. Comparison of the considered Automated Market Maker's invariants.

The considered invariants, due to the uniqueness of x and y for a fixed K, allow to proceed with convex functions $y = y(x, K)$ and $x = x(y, K)$, expressing the volume of one asset through another. For example, for **Uniswap** and **Balancer**, such functions are expressed as

$$y_u(x, K) = \frac{K^2}{x}, \qquad (6)$$

$$y_b(x, K) = \frac{K^{\frac{w_x}{w_y}+1}}{x^{\frac{w_x}{w_y}}}. \qquad (7)$$

The first derivative of $y = y(x, K)$. For a fixed liquidity parameter, K can be written as

$$p(x, K) = -\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}, \qquad (8)$$

which, in the literature [3], is called **marginal price**, because it defines the instantaneous price of asset x in units of y in the automated market-making system. At any AMM, a marginal price changes with every trade, which is called [3, 17] **price slippage**.

The function $I(x, y)$ (or $F(x, y, K)$) is meant to be conservative, and the traded volume is calculated based on the property of its conservation during the trading process. For example, the sent volume Δx and received volume Δy satisfy the equation

$$F(x_0, y_0, K_0) = F\left(x_0 + \gamma_1 \Delta x, y_0 - \frac{\Delta y}{\gamma_2}, K_0\right), \qquad (9)$$

where $\gamma_i = 1 - \phi_i$ and ϕ_1, ϕ_2 are fee rates for sent and received assets, respectively.

For example, in the generally accepted open-source implementations of the **Uniswap** and **Balancer** invariants, $\phi_1 > 0$, $\phi_2 = 0$ are used. However, at **Curve** the fees are different ($\phi_1 = 0$ and $\phi_2 > 0$). The introduction of two types of rates allows for reducting the theoretical analysis to a general form, but at the time of publication, the authors are not aware of variations of existing automated market makers that introduce both $\phi_1 \neq 0$ and $\phi_2 \neq 0$.

Visualisation of the mechanics of trading an asset Δx for Δy with two-sided constant fee rates is shown in Figure 2. The trade occurs from the initial state $O = (x_0, y_0)$ to the final state $E = (x_0 + \Delta x, y_0 - \Delta y)$.

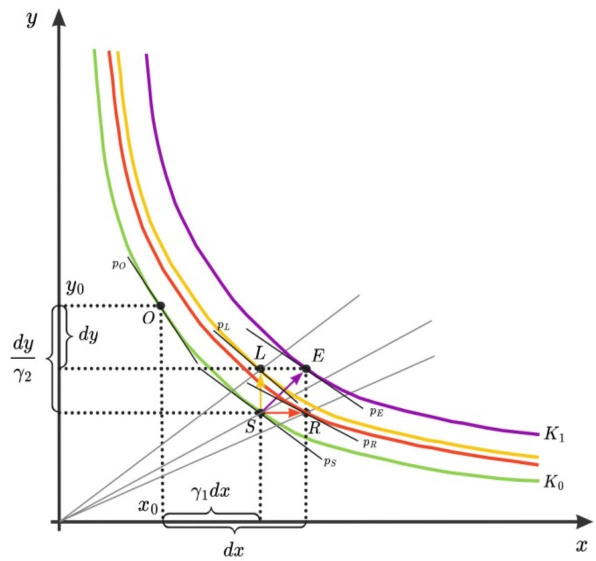

Figure 2. AMM exchange mechanics.

In general, a trade can be divided into two consecutive stages. The first one is the movement from point O to point S preserving the conservative function (8). Since the liquidity parameter K_0 does not change during this stage, then the volume of Δy can be written in integral form using the differential definition of marginal price (7) as

$$\Delta y = \gamma_2 \int_{x_0}^{x_0 + \gamma_1 \Delta x} p(x, K_0) dx. \qquad (10)$$

In the next stage, the earned commissions add up as additional asset reserves, which lead to a subsequent increase in liquidity from K_0 to K_1. The two described stages occur atomically within a single trading transaction [2].

In the particular case when $\gamma_1 \neq 1$ and $\gamma_2 = 1$, the final state of the trading process is at point L. Based on the homogeneity condition (2), we can easily conclude that for the marginal price observed $p_L > p_S$. In the other case ($\gamma_1 = 1$ and $\gamma_2 \neq 1$) $p_R < p_S$. The most general case ($\gamma_1 \neq 1$ and $\gamma_2 \neq 1$) is shown in Figure 2 as point E, for which the ratio of marginal prices p_E and p_S cannot be unambiguously determined.

2.2. Impermanent Loss

In the previous subsection, it was shown how price slippage occurs due to the convex properties of the function $y(x, K)$. Based on this, the concept of **impermanent loss** (**divergence loss**) [3, 7, 10] was introduced, which describes the losses of liquidity providers while external users are trading assets at AMMs. As previously noted, many contemporary articles overlook the fact that while trading at AMMs with a homogeneous conservative function (e.g., **Uniswap**, **Balancer**, **Curve**), the liquidity factor K increases, which, as we will see later, will greatly affect the result. Without taking these changes into account, impermanent losses will always be positive, which means that for any sent volume $\Delta x > 0$, there will be losses in the liquidity pool [3].

To illustrate impermanent losses, two liquidity provider behaviour strategies are examined. The first strategy involves **depositing** assets following the previously outlined trading process. The second strategy entails **passively holding** the initial volumes (x, y) outside the AMM system, assuming the final price resulting from the deposit strategy is observed on the market. Thus, the comparison highlights the impact of actively investing funds in an AMM versus taking a passive approach. For a single trade $\Delta x \ \rightarrow \Delta y$ with respective change in the liquidity $K_0 \rightarrow K_1$ the expressions of the strategies' values V_{hold} and $V_{deposit}$ are represented as

$$V_{deposit} = p(x_0 + \Delta x, K_1) \cdot (x_0 + \Delta x) + (y_0 - \Delta y)$$

$$V_{hold} = p(x_0 + \Delta x, K_1) \cdot x_0 - y_0$$

Next, the absolute and relative impermanent losses are expressed, respectively,

$$IL_a = V_{deposit} - V_{hold} = p(x_0 + \Delta x, K_1) \cdot \Delta x - \Delta y, \quad (11)$$

$$IL_r = \frac{V_{deposit}}{V_{hold}} - 1 = \frac{p(x_0 + \Delta x, K_1) \cdot \Delta x - \Delta y}{p(x_0 + \Delta x, K_1) \cdot x_0 + y_0}. \quad (12)$$

Negative values for (11) and (12) are interpreted as impermanent losses, while positive ones represent **impermanent gains**. These changes are called 'impermanent' because if the marginal price $p(x, k)$ returns to its initial value followed by a sequence of trades within the AMM, the changes vanish. Nevertheless, the market does not always return to its starting price level; therefore, in many market scenarios, it is important to be able to evaluate immediate fluctuations.

3. Impact of Exchange Fee on Impermanent Losses

Suppose the first asset of volume Δx is traded through studied AMM. When fees are involved, the invariant changes its value from $K_0 \rightarrow K_1$ after the trade (see Figure 2). Then, the absolute impermanent loss (11) can be expressed using (10) as

$$IL_a = p(x_0 + \Delta x, K_1)\Delta x - \gamma_2 \int_{x_0}^{x_0 + \gamma_1 \Delta x} p(x, K_0)dx. \quad (13)$$

We utilise the property of the monotonical decreasity of the function $p(x, K)$ (8) with respect to the argument x, which follows from the convexity and decreasity of $y(x, K)$, and estimate the value of losses:

$$IL_a \leq p(x_0 + \Delta x, K_1)\Delta x - \gamma_1\gamma_2 p(x_0 + \gamma_1 \Delta x, K_0)\Delta x,$$

$$IL_a \leq (1 - \gamma_1\gamma_2)p(x_0 + \gamma_1 \Delta x, K_0)\Delta x,$$

from which we get that in the absence of fees ($\gamma_1\gamma_2 = 1$) the right-hand side is equal to 0, therefore, for any trade of volume Δx, there will always be positive impermanent losses for liquidity providers.

Next, we find explicit formulas for IL_a and IL_r in special cases and provide numerical simulations for considered swap invariants.

3.1. Uniswap

Let us substitute the explicit formulas (3) and (6) for **Uniswap** invariant into the expression of the absolute impermanent losses IL_a (13), we obtain

$$IL_a^u = \frac{K_1^2}{(x_0 + \Delta x)^2}\Delta x - \gamma_2 \left(\frac{K_0^2}{x_0} - \frac{K_0^2}{x_0 + \gamma_1 \Delta x}\right).$$

Considering that the final amount of asset y in pool after the single trade is $y_1 = y_0 - \gamma_2 \left(y_0 - \frac{x_0 y_0}{x_0 + \gamma_1 \Delta x}\right)$, the previous equation can be simplified as

$$IL_a^u = \left(y_0 - \gamma_2 \left(y_0 - \frac{x_0 y_0}{x_0 + \gamma_1 \Delta x}\right)\right)\frac{\Delta x}{(x_0 + \Delta x)} - \gamma_2 \left(y_0 - \frac{x_0 y_0}{x_0 + \gamma_1 \Delta x}\right).$$

Introducing the parameter of the relative volume of exchange $\alpha = \frac{\Delta x}{x_0}$, impermanent changes (losses and gains) are expressed as

$$IL_a^u = y_0 \frac{(1 - \gamma_2)(1 + \gamma_1 \alpha) + \gamma_2}{1 + \gamma_1 \alpha} \frac{\alpha}{1 + \alpha} - y_0 \frac{\gamma_1 \gamma_2 \alpha}{1 + \gamma_1 \alpha},$$

or

$$IL^u_a = y_0 \frac{\alpha}{1+\alpha}\frac{1-\gamma_1\gamma_2+\gamma_1(1-2\gamma_2)\alpha}{1+\gamma_1\alpha}. \quad (14)$$

Similarly, considering the relative losses (12), we obtain

$$IL^u_r = \frac{V_{deposit}}{V_{hold}} - 1 = \frac{\alpha(1-\gamma_1\gamma_2+\gamma_1(1-2\gamma_2)\alpha)}{\gamma_2+(1+\gamma_1\alpha)(2-\gamma_2+\alpha)}. \quad (15)$$

The visualisation of IL^u_r depending on the trading fees ϕ_1 and ϕ_2 is presented in Figure 3. Empirically, it is observed that whenever fees are non-zero ($\phi_1+\phi_2 \neq 0$), there exists a region of impermanent gain, which expands along the α-axis as the total sum of fees increases. This effect is further illustrated in Figure 4.

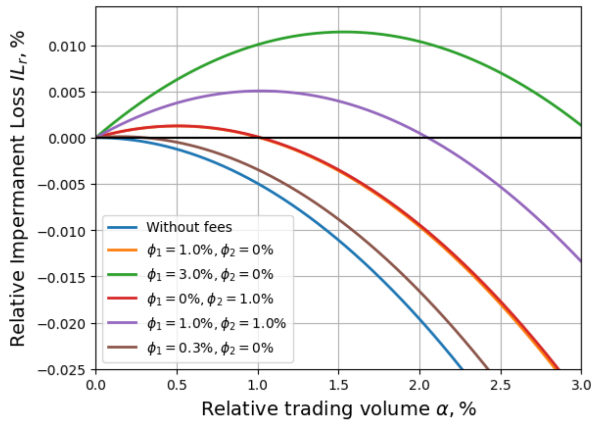

Figure 3. Relative impermanent losses depending on varying trading fees for **Uniswap**.

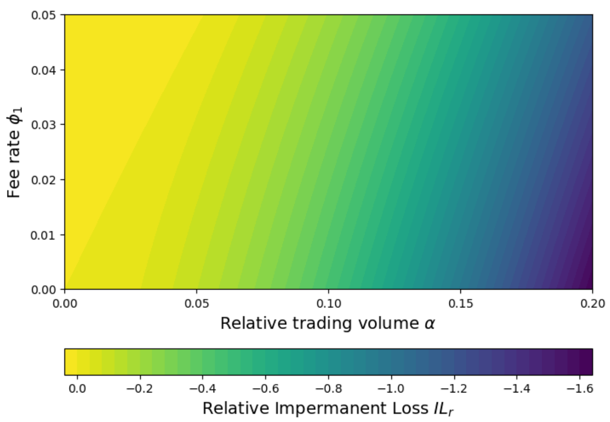

Figure 4. Dependence of IL^u_r on the trading fee parameter ϕ_1 and relative trading volume α for **Uniswap**. The fee rate for the received asset is set to zero ($\phi_2 = 0\%$).

Next, we express the roots of the equation $IL^u_r = 0$ (14) to analytically determine the range $0 < \alpha < \alpha_{max}$, of impermanent gain for **Uniswap**:

$$\alpha_{max} = \frac{1-\gamma_1\gamma_2}{\gamma_1(2\gamma_2-1)}$$

In the limit of small fees $\phi_i \ll 1$, the scale of α_{max} can be estimated using the Taylor expansion:

$$\boldsymbol{\alpha_{max} = \frac{\Delta x_{max}}{x} = (\phi_1+\phi_2+o(\phi_1+\phi_2)) \approx \phi_1+\phi_2,}$$

which is illustrated in Figure 3 or Figure 4 where the boundary of the impermanent gain range reveals an almost linear dependency on the sum of fee rates for **Uniswap**.

3.2. Balancer

Let us use the explicit relations (4) and (7) for the **Balancer**, similar to the previous case of **Uniswap**. The impermanent losses are expressed as

$$IL^b_a = \frac{w_x}{w_y}\frac{K_1^{\frac{w_x}{w_y}+1}}{(x_0+\Delta x)^{\frac{w_x}{w_y}+1}}\Delta x - \gamma_2\left(\frac{K_0^{\frac{w_x}{w_y}+1}}{x_0^{\frac{w_x}{w_y}}} - \frac{K_0^{\frac{w_x}{w_y}+1}}{(x_0+\gamma_1\Delta x)^{\frac{w_x}{w_y}}}\right).$$

By substituting the final volume of an asset $y_1 = y_0 - \gamma_2\left(y_0 - \frac{x_0^{\frac{w_x}{w_y}}y_0}{(x_0+\gamma_1\Delta x)^{\frac{w_x}{w_y}}}\right)$ into the expression for liquidity K_1, we obtain

$$IL^b_a = \frac{w_x}{w_y}\left(y_0 - \gamma_2\left(y_0 - \frac{x_0^{\frac{w_x}{w_y}}y_0}{(x_0+\gamma_1\Delta x)^{\frac{w_x}{w_y}}}\right)\right)\frac{\Delta x}{x_0+\Delta x} - \gamma_2 y_0\left(1 - \frac{x_0^{\frac{w_x}{w_y}}}{(x_0+\gamma_1\Delta x)^{\frac{w_x}{w_y}}}\right).$$

Using the parameter α, one rewrites it as

$$IL^b_a = y_0\frac{w_x}{w_y}\frac{(1-\gamma_2)(1+\gamma_1\alpha)^{\frac{w_x}{w_y}}+\gamma_2}{(1+\gamma_1\alpha)^{\frac{w_x}{w_y}}}\frac{\alpha}{1+\alpha} - \gamma_2 y_0\left(1-\frac{1}{(1+\gamma_1\alpha)^{\frac{w_x}{w_y}}}\right). \quad (16)$$

From relation (16), the derived implicit formula for impermanent losses of **Balancer** conservative function during a single trade is:

$$IL^b_a = y_0\frac{\frac{w_x}{w_y}\left((1-\gamma_2)(1+\gamma_1\alpha)^{\frac{w_x}{w_y}}+\gamma_2\right)\alpha - \gamma_2(1+\alpha)\left((1+\gamma_1\alpha)^{\frac{w_x}{w_y}}-1\right)}{(1+\gamma_1\alpha)^{\frac{w_x}{w_y}}(1+\alpha)}.$$

Substituting the special case $\frac{w_x}{w_y} = 1$, where **Balancer** becomes **Uniswap**, we get (14). For relative impermanent losses, one can derive the following dependence on α:

$$IL_r^b = \\ = \frac{\frac{w_x}{w_y}\left((1-\gamma_2)(1+\gamma_1\alpha)^{\frac{w_x}{w_y}}+\gamma_2\right)\alpha-\gamma_2(1+\alpha)\left((1+\gamma_1\alpha)^{\frac{w_x}{w_y}}-1\right)}{\frac{w_x}{w_y}\left((1-\gamma_2)(1+\gamma_1\alpha)^{\frac{w_x}{w_y}}+\gamma_2\right)+(1+\alpha)(1+\gamma_1\alpha)^{\frac{w_x}{w_y}}}. \quad (17)$$

It is notable that for **Uniswap** and **Balancer**, in accordance with the derived explicit relationships (15) and (17), the impermanent change does not depend on the ratio $\frac{y_0}{x_0}$ when x_0 is fixed.

The visualisation of IL_r^u depending on the trading fees ϕ_1 and ϕ_2 is shown in Figure 5. The results with a fixed parameter $\frac{w_x}{w_y}$ are similar to the ones in Figure 3. The dependence on varying asset weights is presented in Figure 6, showing that the range of impermanent gains shrinks as the ratio $\frac{w_x}{w_y}$ increases. This monotonic behaviour is also demonstrated in the colour plot in Figure 7.

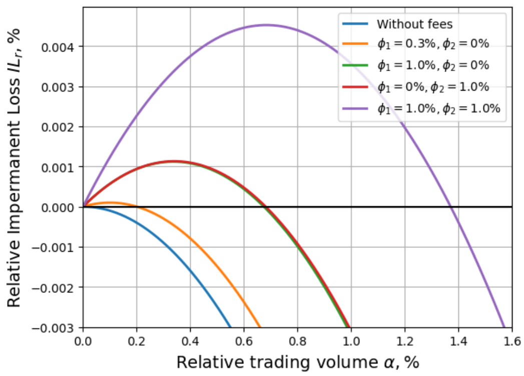

Figure 5. Relative impermanent losses depending on different trading fees for **Balancer** with $\frac{w_x}{w_y} = 2$.

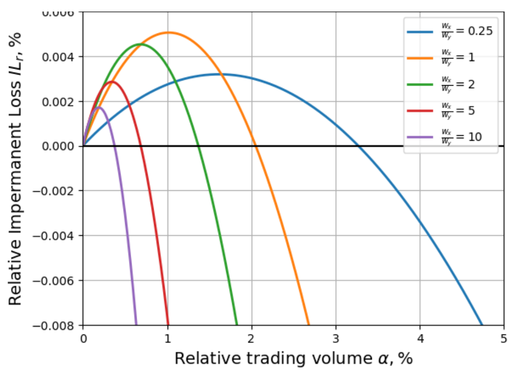

Figure 6. Dependence of **Balancer**'s IL_r^b on parameters of relative asset weights $\frac{w_x}{w_y}$ with fees $\phi_1 = \phi_2 = 1\%$.

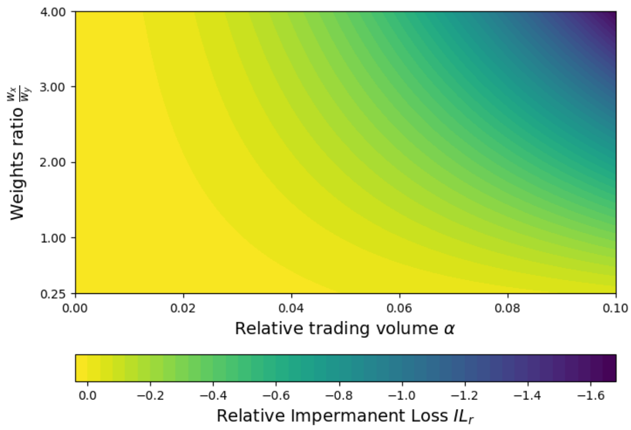

Figure 7. Colour plot of **Balancer**'s relative impermanent losses (17) dependent on trading volume α and reserve ratio $\frac{w_x}{w_y}$. Fixed trading fees $\phi_1 = 3\%$ and $\phi_2 = 0\%$ are implied.

3.3. Curve

Due to the complexity of the **Curve** conservative function (5), it is either cumbersome or even not possible to get explicit formulas similar to **Uniswap** (15) or **Balancer** (17). Hence, numerical simulations were conducted for this case. Figure 8 illustrates how the amplification parameter A influences the magnitude of relative impermanent loss (or gain).

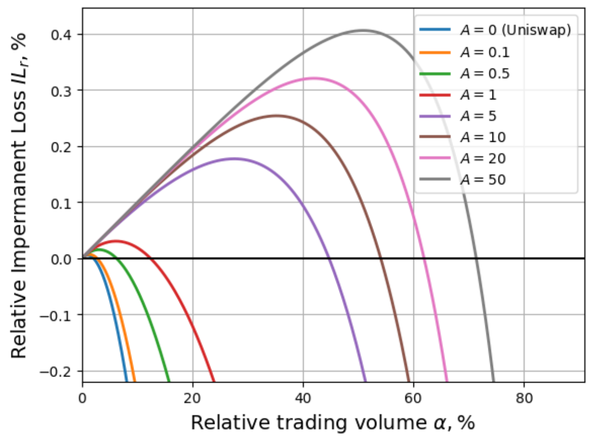

Figure 8. **Curve** IL_r (%) as a function of the relative trade volume α (%) for various amplification parameters A, $\phi_1 = 1\%$, $\phi_2 = 1\%$ and $x_0 = 1, y_0 = 1$.

Contrary to Uniswap (14) and Balancer (16), impermanent loss for Curve conservative function (5) depends on the initial (before the trade) asset ratio $\frac{y_0}{x_0}$. The dependence of the numeric simulation of impermanent losses and gains on the initial reserve ratio $\frac{y_0}{x_0}$ and the relative trading volume $\alpha = \frac{\Delta x}{x_0}$, as depicted in Figure 9, represents a more complex structure for regions of impermanent gain than the corresponding dependencies for **Uniswap** and **Balancer**. The interpretation of this effect is that for **Curve** there is observed relatively slight price slippage [12] for small trades, therefore, trades do not significantly impact the liquidity provider's values (in the assumption of a small price change), unlike the **Uniswap** and **Balancer** invariants, in which the slippage effect is more extreme. In the case of large trades, the trend is different – there is a large price slippage resulting in large losses.

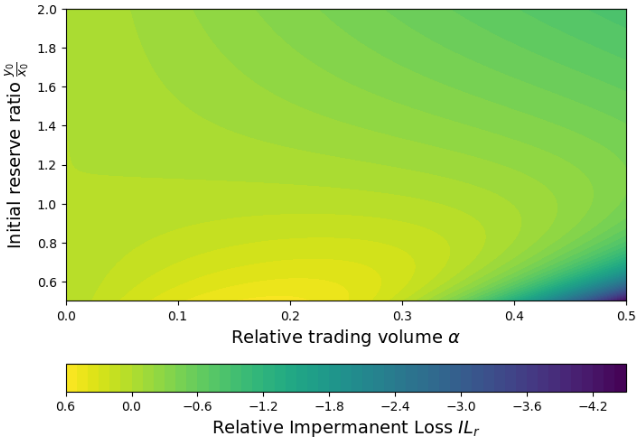

Figure 9. Numeric simulation of IL_r (%) for **Curve** swap invariant as a dependence on the reserve ratio $\frac{y_0}{x_0}$ and α for $A = 5$, $\phi_1 = 0\%$, $\phi_2 = 1\%$ and $x_0 = 1$.

3.4. Approximate Estimation for an Arbitrary Conservative Function

Let's consider the expression for impermanent losses (11) using the Taylor series to the second order:

$$IL_a = \left(p(x_0, K_0) + \frac{\partial p}{\partial x}(x_0, y_0)\Delta x + \frac{\partial p}{\partial K}(x_0, y_0)\Delta K + o(\Delta x) + o(\Delta K) \right)\Delta x - \gamma_1\gamma_2 p(x_0, K_0)\Delta x - \frac{1}{2}\gamma_1^2\gamma_2 \frac{\partial p}{\partial x}(x_0, y_0)\Delta x^2 + o(\Delta x^2).$$

From previous calculations, neglecting higher-order's terms and using the results from **Appendix A**, one can obtain the root of the quadratic equation describing the region of impermanent gain:

$$\Delta x_{max} \approx \frac{(1-\gamma_1\gamma_2)p(x_0, K_0)}{\left(1 - \frac{1}{2}\gamma_1^2\gamma_2\right)\frac{\partial p}{\partial x} - (1-\gamma_1\gamma_2)\cdot\frac{\partial p}{\partial K}\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial K}}}.$$

Recall that $\gamma_i = 1 - \phi_i$ and in practice $\phi_i \ll 1$, then

$$\Delta x_{max} \approx 2(\phi_1 + \phi_2)\frac{p(x_0, K_0)}{\left|\frac{\partial p(x_0, K_0)}{\partial x}\right|}. \quad (18)$$

So, to ensure positive impermanent gain, the estimate of the relative price change $\frac{\left|\frac{\partial p(x_0,K_0)}{\partial x}\right|\Delta x}{p(x_0, K_0)}$ in the trade should be approximately no greater than twice the sum of the fee rates of AMM. Ultimately, within small trading volumes, the expression (18) allows for quick calculation of the scales of single trade which won't lead to impermanent losses for liquidity providers.

3.5. Invariant Comparison

A comparison of the dependencies of impermanent losses for **Uniswap**, **Balancer** and **Curve** invariants is shown in Figure 10. The values from **Appendix B** were chosen as the parameters for the simulation. It is important that for all considered invariants with proposed parameters, there is a region of impermanent gain for values of relative trade volumes $\alpha < 0.2\%$. The largest region of impermanent gain is achieved for the hybrid invariant **Curve** with $\frac{y_0}{x_0} = 1$, for which the fee rate takes minimum relative values (0.01% and 0.02%, respectively). In this case, the amplification coefficient $A = 10$ or $A = 15$, so the behaviour is very different from **Uniswap**, similar to the dependence shown in Figure 8.

Another way to get similar conclusions is to use the approximation in Eq. (18). It was found that the size of the gain region is inversely proportional to the initial derivative of price during the exchange, which for **Curve** invariant (Figure 1) with $\frac{y_0}{x_0} = 1$ has a smaller value than for **Uniswap** and **Balancer**.

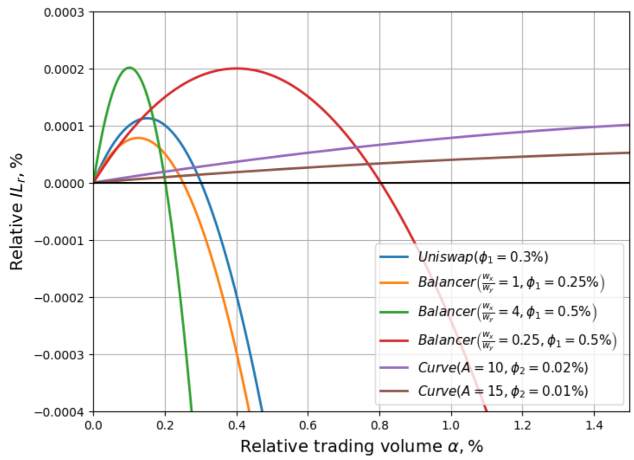

Figure 10. Comparison of relative impermanent changes IL_r for different swap invariants with parameters from **Appendix B**. For **Curve**, we additionally suppose that $\frac{y_0}{x_0} = 1$.

4. Practical Analysis on Real-World Data

In this section, a naive method of adjusting AMM fees will be proposed based on the requirement to optimise the area of impermanent gain during a single trade for liquidity providers. The invariants listed in **Appendix B** will be selected as real data sources for collecting statistics from January 2024 to December 2024.

4.1. Data Analysis

The statistical analysis of the obtained data is shown in Table 1. For the **Balancer** pool wstETH/AAVE, the weight ratio parameter $\frac{w_x}{w_y}$ depends on the direction of the swap, therefore two different values are chosen for this case.

AMM pool	Swap direction	α_{mean}, %	α_{median}, %	α_{99}, %
Uniswap WETH/USDC	Both	0.007	0.001	0.094
Uniswap WETH/USDT	Both	0.005	0.001	0.073
Uniswap WETH/WBTC	Both	0.034	0.003	0.333
Sushiswap WETH/USDC	Both	0.023	0.005	0.231
Sushiswap WETH/DAI	Both	0.086	0.063	0.502
Sushiswap WETH/WBTC	Both	0.031	0.002	0.234
Balancer WETH/WBTC	Both	0.124	0.088	0.666
Balancer wstETH/AAVE	>	0.094	0.075	0.427
	<	0.022	0.018	0.083
Curve FRAX/USDC	Both	1.032	0.1852	10.188
Curve frxETH/WETH	Both	0.313	0.0328	4.6295

Table 1. Statistical analysis of the relative trade volume α of selected AMM pools. The mean α_{mean}, median α_{median} and 99th percentile α_{99} were chosen as the main values described in the distribution.

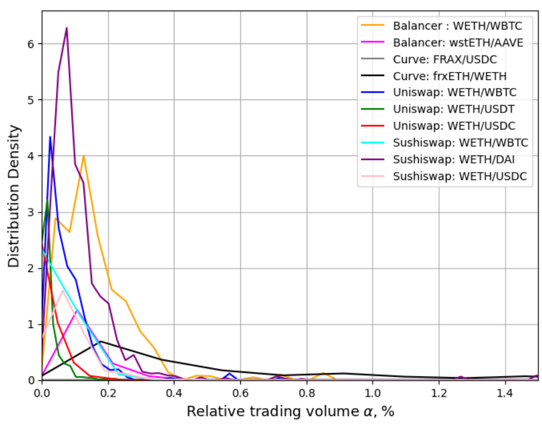

Figure 11. Comparison of the distributions of relative trade volume α for different AMM pools from **Appendix B.**

The histogram distributions of the trade volume α are shown in Figure 11. It can be concluded that for **Uniswap**, **Sushiswap** (fork of **Uniswap**) and **Balancer** pools the most concentrated sizes of swap are within $\alpha < 0.3\%$.

Analysis of the share of total trade number exhibits the impermanent gain effect described in the theoretical section. Table 2 shows that for most of the pools of **Uniswap**, **Sushiswap** and **Balancer,** this value is greater than 95%. However, this is not true for **Curve**, since the distribution in Figure 11 is dominated by large volumes α. The proposed values of 44.64% and 52.89% do not contradict with the results in Figure 11, since the calculation in Table 2 did not assume that $\frac{y_0}{x_0} = 1$, and the instantaneous real values of reserves were taken from the pools.

AMM pool	Swap direction	% of swaps with impermanent gain effect
Uniswap WETH/USDC	Both	99.88
Uniswap WETH/USDT	Both	99.93
Uniswap WETH/WBTC	Both	98.86
Sushiswap WETH/USDC	Both	99.43
Sushiswap WETH/DAI	Both	97.51
Sushiswap WETH/WBTC	Both	99.39
Balancer WETH/WBTC	Both	93.20
Balancer wstETH/AAVE	>	95.13
	<	95.82
Curve FRAX/USDC	Both	52.89
Curve frxETH/WETH	Both	44.64

Table 2. The share of trades that lead to the effect of impermanent gain for liquidity providers.

4.2. Trade Fee Rate Adjustment

As a practical recommendation, we assume the simplest way to manage the level of fee rate based on historical trading statistics. For this, we use a naive approach, according to which the rate on day T will be selected from the condition that $Q\%$ trades of the previous week $T-7, \ldots, T-1$ should not have impermanent losses (i.e., the presence of impermanent gain). Visualisation of this mechanism for adjusting fee rate is shown in Figure 12 (for WETH/Stablecoin pairs) and Figure 13 (for WETH/WBTC pairs), where $Q = 99\%$. It appears that for different pools, the proposed fee rate differs from the constant value of 0.3%. WETH/WBTC pairs are more volatile, so this method of adjustment leads to higher fee values than in the case of pairs with stable tokens. This observation is confirmed by the results of the statistical study in Table 2. Pairs WETH/USDC, WETH/DAI (Sushiswap, Figure 14), and WETH/WBTC (Balancer, Figure 12) are outstanding, which is associated with the fact that the trading liquidity and volumes of these pairs are smaller than the others.

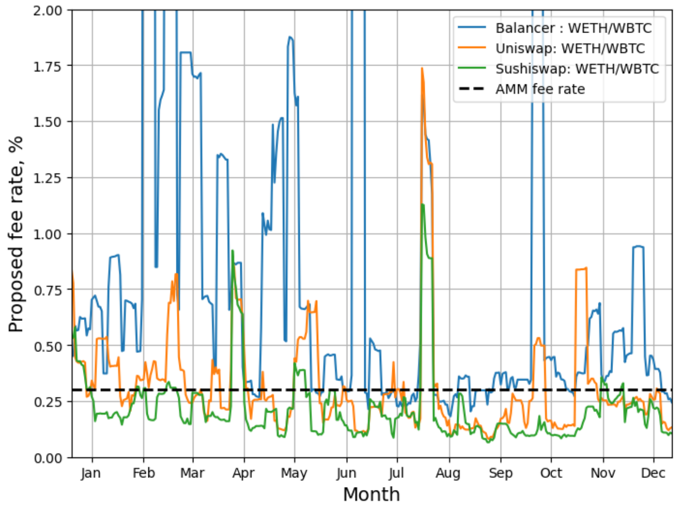

Figure 12. Results for adjusted exchange fee rate based on historical trades of the previous week for WETH/WBTC pools with Q=99%.

Let us consider the influence of the fee adjustment in more detail for the popular WETH/USDC pool with **Uniswap** invariant. In Figure 13, the dependencies are shown for different Q values in comparison with average price and trading volumes. These results suggest that more activity on the market leads to a higher proposed fee (for retaining the impermanent gain effect). This conclusion makes sense since such situations often happen as a result of news events and increased fees boost the profitability of liquidity providers. This observation illustrates the potential for non-trivial dynamic fee rate management even for classical AMM invariants.

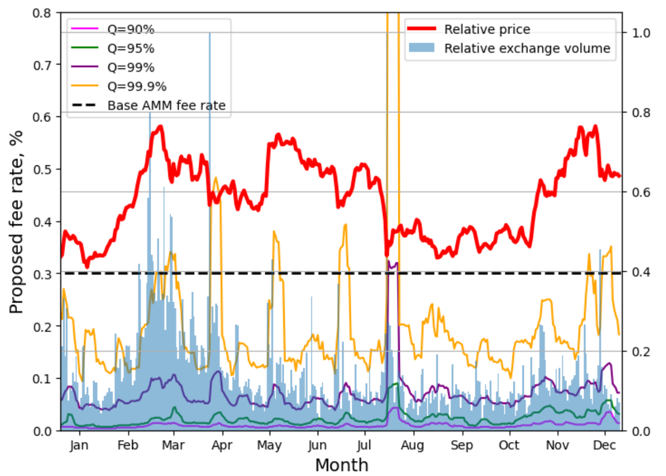

Figure 13. Results for adjusted exchange fee rate for WETH/USDC pool at different Q percentile values compared to price and trading volumes (shown in relative values).

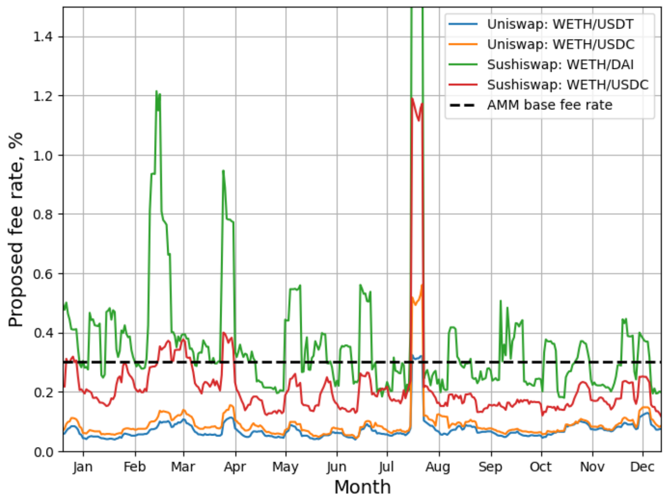

Figure 14. Results for adjusted exchange fee rate based on historical trades of the previous week for WETH/Stablecoin pools with Q=99%.

5. Limitations

The focus on the study of the single swap impact in an AMM on liquidity providers may lead to some distortion in the conclusions regarding the real dynamics of the market situation. Impermanent changes are generally affected by the arbitrary sequence of trades. Therefore, the obtained formulas for impermanent losses and gains could be considered only as upper estimates for the corresponding values.

In addition, the proposed study does not consider complex pricing dependencies on many external factors, such as volatility or changes in the overall liquidity of assets over time. In real market conditions, trading involves taking into consideration price fluctuations, various strategies of participants (high-frequency and positional trading, arbitrage and other types), as well as randomness in making decisions on transactions, which can significantly affect the actual costs for liquidity providers for a period of provisioning.

Furthermore, the optimal fee rate is determined by the exchange's attractiveness. Increased fees could diminish trading volumes, potentially leading to higher overall costs. This underscores the significance of complex and thorough evaluation of the variables affecting impermanent changes.

6. Conclusion

Throughout our investigation of the influence of the swap fee parameter on the impermanent losses of liquidity providers in AMM systems like **Uniswap**, **Balancer** and **Curve**, overall significant conclusions can be stated.

Firstly, we discovered that the trading fee parameters inherent in AMM mechanisms exert a direct effect on the magnitude of impermanent losses incurred by liquidity providers.

Specifically, during small trades ($\alpha < 0.2\%$ for all considered pools), these losses frequently turn out to be impermanent gains. Moreover, for the majority of the studied pools of the **Uniswap**, **Sushiswap** and **Balancer** protocols, it turned out that 95% of historical single swaps lead to impermanent gain. This conclusion could be insightful for novel approaches to optimise liquidity management strategies, reduce risks and increase profitability.

Secondly, the theoretical formulas developed throughout this study serve as valuable instruments for analysing and predicting liquidity strategies depending on changes in fees on a microeconomic level. The obtained dependencies are applicable not only to practitioners but also to researchers seeking to assess the potential consequences of various fee management scenarios. For example, the proposed naive way to set fee rates based on weekly historical data could be used to optimise the gain of liquidity providers from a single trade (and presumably to optimise the integral return). The authors consider a more detailed analysis of this approach and methods of backtesting to be the next stage of researching this topic.

Additionally, our findings highlight the necessity for ongoing analysis regarding the fee structures for AMM protocols with conservative invariant functions. By refining our understanding of how different fee levels impact liquidity provision, we can enhance the efficiency and stability of decentralised exchanges, thereby contributing to the broader ecosystem's growth and sustainability. Ultimately, this research underscores the importance of comprehensive and fundamental studies focusing on the investigation of swap fees and liquidity dynamics.

Competing Interests:
None declared.

Ethical Approval:
Not applicable.

Author's Contribution:
VR: main author, methodology, theoretical investigation, data collection and validation, writing original manuscript. VG: scientific supervision, methodology, proofreading and editing, discussion. AB: scientific supervision, proofreading and editing, discussion, work presentation.

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Appendix A. Differentials in the Exchange Process

Among the set of variables (x, y, K), only one is independent due to the presence of two binding relations—the implicit expression of the invariant (1) and the exchange equation for $\Delta x \rightarrow \Delta y$ (9). Using the smoothness of these functions, we rewrite them in differential form:

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial K} dK = 0$$

and

$$dy = -\gamma_1 \gamma_2 \cdot p(x, K) \cdot dx = -\gamma_1 \gamma_2 \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} dx.$$

Substituting one expression into another, we get

$$\frac{\partial F}{\partial x} dx - \gamma_1 \gamma_2 \frac{\partial F}{\partial y} \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} dx + \frac{\partial F}{\partial K} dK = 0,$$

from which we obtain the desired relationship of differentials

$$dK = (1 - \gamma_1 \gamma_2) \left(- \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial K}} \right) dx.$$

Appendix B. Ethereum AMM Pools

In the study, the Ethereum data was parsed for the period from January 2024 to December 2024. The addresses of the specific pools **Uniswap v2**, **Sushiswap v2** (fork of **Uniswap v2**), **Balancer v2** and **Curve** are listed in Tables 3–6.

Pool	Ethereum Address
WETH/USDC	0xb4e16d0168e52d35cacd2c6185b44281ec28c9dc
WETH/USDT	0x0d4a11d5eeaac28ec3f61d100daf4d40471f1852
WETH/WBTC	0xbb2b8038a1640196fbe3e38816f3e67cba72d940

Table 3. Addresses of selected **Uniswap v2** pools.

Pool	Ethereum Address
WETH/USDC	0x397ff1542f962076d0bfe58ea045ffa2d347aca0
WETH/USDT	0xc3d03e4f041fd4cd388c549ee2a29a9e5075882f
WETH/WBTC	0xceff51756c56ceffca006cd410b03ffc46dd3a58

Table 4. Addresses of selected **Sushiswap v2** pools.

Pool	Ethereum Address
WETH/WBTC	0xa6f548df93de924d73be7d25dc02554c6bd66db5
wstETH/AAVE	0x3de27efa2f1aa663ae5d458857e731c129069f29

Table 5. Addresses of selected **Balancer v2** pools.

Pool	Ethereum Address
FRAX/USDC	0xdcef968d416a41cdac0ed8702fac8128a64241a2
frxETH/WETH	0x9c3b46c0ceb5b9e304fcd6d88fc50f7dd24b31bc

Table 6. Addresses of selected **Curve** pools.

It is important to note that **Curve** pools are used for stable pairs, often contain more than two tokens and use price oracle data, therefore, only pools from the Frax Finance ecosystem were selected because they match our theoretical description.

Table 7 is a summary table of the invariant parameters used for the pools in the study. Since **Curve** AMM is used for relatively stable pairs, then fee rates are about 10–50 times smaller than in other pools.

AMM type	AMM Pool	Fee rate (%)	Additional parameters
Uniswap	WETH/USDC	$\phi_1 = 0.3$	-
	WETH/USDT	$\phi_1 = 0.3$	-
	WETH/WBTC	$\phi_1 = 0.3$	-
Sushiswap	WETH/USDC	$\phi_1 = 0.3$	-
	WETH/DAI	$\phi_1 = 0.3$	-
	WETH/WBTC	$\phi_1 = 0.3$	-
Balancer	WETH/WBTC	$\phi_1 = 0.25$	$\frac{w_{WETH}}{w_{WBTC}} = 1$
	wstETH/AAVE	$\phi_1 = 0.5$	$\frac{w_{wstETH}}{w_{AAVE}} = \frac{1}{4}$
Curve	FRAX/USDC	$\phi_2 = 0.01$	$A = 15$
	frxETH/WETH	$\phi_2 = 0.02$	$A = 10$

Table 7. Fee rates and specific parameters of AMM pools studied in the current work.